# A regime-switching model with jumps and its application to bond pricing and insurance

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In this paper, we consider a Markovian, regime-switching model with jumps and its application to bond pricing and insurance. The jumps in the model are described by a compound Cox process, where the arrival intensity of the counting number-process follows a regime-switching shot noise process. Using a martingale method, we derive exponential-affine form expressions for the price of a zero-coupon bond and the joint Laplace transform of the aggregate accumulated claims and the arrival intensity.

*Key words:* Markov chain; regime-switching shot noise process; zero-coupon bond; aggregate accumulated claims

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## 1. Introduction

In recent years, modelling the dynamics of short rates has long been an interesting and challenging issue in economics and finance. In the past few decades or so, various quantitative models have been proposed to model short rates in the academic literature. Examples include [6, 14, 20]. Basically, there are two common features of these models. Firstly, the short rate process has the mean-reverting property, which has been well-documented in the empirical literature. Secondly, the evolution of short rates over time is described by a continuous-time diffusion process. However, in practice, some extraordinary market events may have large economic impact on short rates and cause jumps in short rates. Short rate models based on Brownian information flows can not describe such large movements in short rates. Many researchers considered jump-diffusion processes, or related processes, to incorporate large jumps in short rates, see [1, 7].

In recent years, there is a growing interest in the applications of regime switching models to various financial problems. Regime-switching models are aimed at capturing the appealing idea that the macro-economy is subjected to regular, yet unpredictable in time, regimes, which in turn affect the prices of financial securities. As pointed out in [15], the study of switching diffusions is triggered by the pressing need of models in which both continuous dynamics and discrete events coexist. Similarly, the motivation of the use of switching jump diffusions stems from developing models in which continuous dynamics are replaced by continuous dynamics with Poisson type jumps. The regime-switching models were introduced by Hamilton [13] to financial econometrics and economists. In a regime-switching model, the market is assumed to be in different states depending on the state of the economy. Regime shifts from one economic state to another may occur due to various financial factors like changes in business conditions, management decisions and other macroeconomic conditions. For instance, as explained in [21], in a stock market, the regimes can be roughly divided into two states, bull market and bear market. The market sentiment and reaction to the two states are in stark contrast. Normally, a bear market is more volatile than that of a bull market. Therefore, it is sensible and necessary to incorporate the impact of structural changes in economic conditions into modelling. In an insurance market, regime-switching models can capture the features of insurance policies that are subject to the economic or political environment changes. Some works on the use of regime-switching models in finance and insurance can be found in [4,5, 10, 19, 21, 22].

In this paper, we propose a Markovian regime-switching jump-augmented Vasicek model for the short rate dynamics, where the jump intensity in the short rate is described by a regime-switching shot noise process. We also apply the regime-switching shot noise process to investigate the number-counting process of the claims in insurance. Shot noise processes are one type of pure jump models which allow for explicit solutions of many important quantities in finance, see [8, 16]. They can be well used to measure the impact of major events on the arrival intensities. As pointed out in [8], the shot noise process measures the frequency, magnitude and time period needed to go back to the previous level of intensity immediately after major events occur. In order to incorporate the impact of structural changes in economic conditions or external environment on the jump intensity, this paper extends the shot noise process to a regime-switching version.

The aim of this paper is to provide a Markov, regime-switching model for investigating the pricing of a zero-coupon bond in finance and the expected aggregate accumulated loss in insurance. This paper is organized as follows. Section 2 presents the dynamics of the short rate. In Section 3, we derive the price of a zero-coupon bond. Section 4 considers a compound Cox insurance risk model, where the claim-number process is driven by a Cox process with regime-switching shot noise intensity. Section 5 concludes.

## 2. The model dynamics for the short rate

Consider a continuous-time model with a finite time horizon $\mathcal{T} = [0, T^*]$, where $T^* < \infty$. Let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ be a filtered complete probability space where the filtration $\mathbb{F} := \{\mathcal{F}_t | t \in \mathcal{T}\}$ satisfies the usual conditions and $P$ is a risk-neutral probability measure. Throughout the paper, it is assumed that all random variables and stochastic processes are well defined on this probability space and $\mathcal{F}_{T^*}-$measurable.

Let $\mathbf{X} := \{X(t) | t \in \mathcal{T}\}$ be a homogenous continuous-time, finite-state, irreducible, observable Markov chain on $(\Omega, \mathcal{F}, \mathbb{F}, P)$ with generator $Q = (q_{ij})_{i,j=1,2,\cdots,N}$, describing the evolution of the state of an economy over time. As in [5], the state space of $\mathbf{X}$ can be taken to be, without loss of generality, the set of unit vectors $\mathcal{E} = \{e_1, e_2, \cdots, e_N\}, e_i = (0, \cdots, 0, 1, 0, \cdots, 0)^* \in R^N$, where $*$ denotes the transpose of a vector or a matrix. The states of the chain $\mathbf{X}$ represent different states of an economy or different stages of a business cycle. Let $\mathbb{H}^X := \{\mathcal{H}_t^X | t \in \mathcal{T}\}$ be a right-continuous, $P$-complete, natural filtration generated by the chain $\mathbf{X}$. Elliott et al. [12] provided the following semi-martingale decomposition for the Markov chain:

$$X(t) = X(0) + \int_0^t Q^* X(s) ds + \mathbf{M}(t), \tag{2.1}$$

where $\{\mathbf{M}(t) | t \in \mathcal{T}\}$ is an $R^N$-valued, $(\mathbb{H}^X, P)$ -martingale.

Assume that the notation $\langle ., . \rangle$ denotes the Euclidean scalar product in $R^N$. That is, for any $x, y \in R^N$, $\langle x, y \rangle = \sum_{i=1}^N x_i y_i$.

Consider a real-valued, pure jump process $\mathbf{J} := \{J(t) | t \in \mathcal{T}\}$ on $(\Omega, \mathcal{F}, \mathbb{F}, P)$ such that

$$J(t) = \sum_{j=1}^{N(t)} Y_j, \tag{2.2}$$

where $\mathbf{N} := \{N(t) | t \in \mathcal{T}\}$ is a Cox process with stochastic intensity $\boldsymbol{\lambda} := \{\lambda(t) | t \in \mathcal{T}\}$, the jump amounts $Y_j, j = 1, 2, \cdots$ are assumed to be independent and identically distributed with the common distribution $F$ satisfying some suitable integrability conditions.

Recall that a Cox process, also known as a doubly stochastic Poisson process, is a stochastic process which is a generalization of a Poisson process where the time-dependent intensity is itself a stochastic process. In this paper we assume the stochastic intensity $\boldsymbol{\lambda} := \{\lambda(t)|t \in \mathcal{T}\}$, starting from $\lambda(0) = \langle \overline{\boldsymbol{\lambda}}, X(0)\rangle$ for a constant vector $\overline{\boldsymbol{\lambda}} = (\overline{\lambda}_1, \cdots, \overline{\lambda}_N)^* \in R^N$ with $\overline{\lambda}_i > 0$ for each $i = 1, 2, \cdots, N$, is modelled by

$$d\lambda(t) = -b\lambda(t)dt + dL(t), t \in \mathcal{T}, \tag{2.3}$$

where $b > 0$ is a constant and the process $\mathbf{L} := \{L(t)|t \in \mathcal{T}\}$ with

$$L(t) = \sum_{j=1}^{\overline{N}(t)} Z_j$$

is a regime-switching compound Poisson process. Here $\overline{\mathbf{N}} := \{\overline{N}(t)|t \in \mathcal{T}\}$ is a regime-switching Poisson process with the intensity given by

$$\rho(t) = \langle \boldsymbol{\rho}, X(t)\rangle,$$

for a constant vector $\boldsymbol{\rho} = (\rho_1, \cdots, \rho_N)^* \in R^N$ with $\rho_i > 0$ for each $i = 1, 2, \cdots, N$. That is to say, if $X(s) = e_j$ for all $s$ in a small interval $(t, t+h]$, then $\overline{N}(t+h) - \overline{N}(t)$ has a Poisson distribution with parameter $\rho_j$. The non-negative jump amounts $Z_j, j = 1, 2, \cdots$ are assumed to be independent and identically distributed conditional on $\mathcal{H}^X$. Assume given $X(t)$, the size of the jump that occurs at time $t$ has the conditional distribution $H_t(.) = \langle \mathbf{H}(.), X(t)\rangle$ supported on $(0, \infty)$, where $\mathbf{H}(.) = (H^1(.), \cdots, H^N(.))^* \in R^N$, and the distribution functions $H^i(.), i = 1, 2, \cdots, N$, satisfy some suitable integrability conditions. More precisely, if $X(t) = e_j$, then the size of the jump that occurs at time $t$ has the conditional distribution $H^j$. Furthermore, we assume $\overline{\mathbf{N}}, \mathbf{N}$ and $\mathbf{X}$ do not jump at the same time.

Eq. (2.3) can be rewritten as

$$\lambda(t) = e^{-bt}\lambda(0) + \int_0^t e^{-b(t-u)} dL(u), t \geq 0. \tag{2.4}$$

The intuitive interpretation of (2.4) is that at the each jump time of $\overline{\mathbf{N}}$, the process $\boldsymbol{\lambda}$ jumps upward. Otherwise it decays exponentially with rate $b > 0$, which is consistent with the intuition that the intensity should go back to the previous level after major events occur. However, if $b \leq 0$, then $\lambda(t)$ is an increasing process with $t$, which could not be realistic. The intensity modelled by (2.3) is called a regime-switching shot noise process, which is a generalization of a shot noise process. In the regime-switching shot noise process, the intensity process $\boldsymbol{\lambda}$ can be impacted by the economic conditions described by a Markovian chain.

Now we are ready to model the short rate by a Markov, regime-switching, jump-augmented Vasicek model.

For each $t \in \mathcal{T}$, let $a(t)$ be the level of mean reversion of the short rate process at time $t$. This is also called the central tendency of the short rate. We suppose that $a(t)$ is driven by the state of the chain $\mathbf{X}$ as follows:

$$a(t) = \langle \mathbf{a}, X(t)\rangle,$$

for a constant vector $\mathbf{a} = (a_1, a_2, \cdots, a_N)^*$, with $a_i > 0$ for each $i = 1, 2, \cdots, N$.

For each $t \in \mathcal{T}$, define $\sigma(t)$ to be the volatility rate of the short rate process at time $t$. Suppose that $\sigma(t)$ is also modulated by the state of the chain $\mathbf{X}$ as follows:

$$\sigma(t) = \langle \boldsymbol{\sigma}, X(t)\rangle,$$

for a constant vector $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \cdots, \sigma_N)^*$ with $\sigma_i > 0$ for each $i = 1, 2, \cdots, N$.

Let $c > 0$ be a mean-reversion coefficient, which controls the speed of mean-reversion. Let $\mathbf{W} := \{W(t)|t \in \mathcal{T}\}$ be a standard Brownian motion with respect to its right-continuous, $P$-complete, natural filtration. Assume that the Brownian motion $\mathbf{W}$ is independent of other stochastic processes under the measure $P$.

Then we assume that under the risk-neutral measure $P$, the evolution of the short rate, denoted by $\mathbf{r} := \{r(t)|t \in \mathcal{T}\}$, is governed by the following process:

$$dr(t) = c(a(t) - r(t))dt + \sigma(t)dW(t) + dJ(t), t \in \mathcal{T}, \tag{2.5}$$

where $J(t)$ is defined in (2.2).

Note that if we remove the component $J(t)$ from (2.5), then the short rate follows a Markovian regime-switching Hull-White model. Here the jump process $\mathbf{J}$ describes the impact of some extraordinary market events, such as market crashes, interventions by central banks or monetary authorities, on the short rate.

We now specify the information structure of our model. Let $\mathbb{H}^r := \{\mathcal{H}_t^r|t \in \mathcal{T}\}$ and $\mathbb{H}^\lambda := \{\mathcal{H}_t^\lambda|t \in \mathcal{T}\}$ be the right-continuous, $P$-completed, natural filtrations generated by the processes $\mathbf{r}$ and $\boldsymbol{\lambda}$, respectively. Denote the enlarged filtration by $\mathbb{H} := \{\mathcal{H}_t|t \in \mathcal{T}\}$, where for each $t \in \mathcal{T}$, $\mathcal{H}_t = \mathcal{H}_t^r \vee \mathcal{H}_t^\lambda \vee \mathcal{H}_t^X$, be the minimal $\sigma$-field containing $\mathcal{H}_t^r, \mathcal{H}_t^\lambda$ and $\mathcal{H}_t^X$.

Now we discuss the pricing problems under a Markovian environment. Since $P$ is a risk-neutral probability measure, the price at time $t$ of an $\mathcal{H}_T-$measurable promised payment $\Xi$ is:

$$\gamma(t,T) = E[\Xi e^{-\int_t^T r(s)ds}|\mathcal{H}_t]. \tag{2.6}$$

In this paper, we focus on the case when the promised payment $\Xi$ is of the form $\theta(X(T)) = \langle \boldsymbol{\theta}, X(T)\rangle$ for a vector $\boldsymbol{\theta} = (\theta_1, \cdots, \theta_N)^*$ with $\theta_i > 0$ for each $i = 1, 2, \cdots, N$. In this case, the price at time $t$ of the promised payment $\theta(X(T))$ is given by:

$$\gamma(t,T) = E[\theta(X(T))e^{-\int_t^T r(s)ds}|\mathcal{H}_t]. \tag{2.7}$$

Letting $\theta(X(T)) = 1$, then the price of a zero-coupon bond at time $t$ with maturity $T$ is

$$P(t,T) = E[e^{-\int_t^T r(s)ds}|\mathcal{H}_t]. \tag{2.8}$$

Note that $(\mathbf{r}, \boldsymbol{\lambda}, \mathbf{X})$ is a three-dimensional Markov process with respect to $\mathbb{H}$. Consequently, given that $X(t) = x, r(t) = r, \lambda(t) = \lambda$,

$$\gamma(t,T) = E[\theta(X(T))e^{-\int_t^T r(s)ds}|r(t) = r, \lambda(t) = \lambda, X(t) = x] =: V(t,T,r,\lambda,x). \tag{2.9}$$

In what follows, we suppose that the function $V(t,T,r,\lambda,x)$ is continuously differentiable with respect to $t, \lambda$, and twice continuously differentiable with respect to $r$, where the corresponding partial derivatives are denoted by $\frac{\partial V}{\partial t}, \frac{\partial V}{\partial \lambda}, \frac{\partial V}{\partial r}$ and $\frac{\partial^2 V}{\partial r^2}$, respectively. In the next section, we shall derive the formula for the price $\gamma(t,T)$.

## 3. Pricing formula

In this section, we adopt a martingale approach to derive an exponential affine formula for the price at time $t$ of the promised payment $\theta(X(T))$.

**Theorem 3.1** *Under some suitable integrability conditions, the price at time $t$ of the promised payment $\theta(X(T))$ is given by*

$$\gamma(t,T) = e^{A(t,T)r + B(t,T)\lambda}\langle \Psi(t,T)\boldsymbol{\theta}, x\rangle, \tag{3.1}$$

*where*

$$A(t,T) = \frac{e^{-c(T-t)} - 1}{c}, \tag{3.2}$$

$$B(t,T)=\int_t^T\int_{-\infty}^{\infty}e^{-b(s-t)}(e^{A(s,T)y}-1)dF(y)ds, \tag{3.3}$$

*and* $\Psi(t,T)$ *is a fundamental matrix solution to the following linear, matrix-valued, ODE:*

$$\frac{\partial\Psi(t,T)}{\partial t}+\Delta(t,T)\Psi(t,T)=0, \Psi(T,T)=\mathbf{diag}(\boldsymbol{\theta}), \tag{3.4}$$

*with* $\Delta(t,T)=Q+\mathbf{diag}(\mathbf{G}_t)$*. Here* $\mathbf{diag}(\mathbf{G}_t)$ *is a diagonal matrix, where the diagonal entries are given by the vector* $\mathbf{G}_t=(G_t^1,G_t^2,\cdots,G_t^N)^*\in R^N$*, with*

$$G_t^i=\rho_i\int_0^{\infty}(e^{B(t,T)z}-1)dH^i(z)+cA(t,T)a_i+\frac{\sigma_i^2}{2}A^2(t,T), i=1,\cdots,N. \tag{3.5}$$

*Proof.* Write

$$\mathbf{V}(t,T,r,\lambda)=(V(t,T,r,\lambda,e_1),V(t,T,r,\lambda,e_2),\cdots,V(t,T,r,\lambda,e_N))^*.$$

Hence,

$$V(t,T,r,\lambda,x)=\langle\mathbf{V}(t,T,r,\lambda),x\rangle.$$

It is easy to see that the process

$$U(t,T,r,\lambda,x)=e^{-\int_0^t r(s)ds}V(t,T,r,\lambda,x)=E[\theta(X(T))e^{-\int_t^T r(s)ds}|\mathcal{H}_t]$$

is an $(\mathbb{H},P)-$martingale. Applying Ito's formula to $U(t,T,r,\lambda,x)$ yields

$$\begin{aligned}
&dU(t,T,r,\lambda,x)=-r(t)e^{-\int_0^t r(s)ds}V(t,T,r,\lambda,x)dt+e^{-\int_0^t r(s)ds}\\
\times\ &[\frac{\partial V}{\partial t}dt+\frac{\sigma(t)^2}{2}\frac{\partial^2 V}{\partial r^2}dt+\frac{\partial V}{\partial r}(c(a(t)-r)dt+\sigma(t)dW(t))-\frac{\partial V}{\partial\lambda}b\lambda dt\\
+\ &\langle\mathbf{V}(t,T,r,\lambda),Q^*x\rangle dt+\langle\mathbf{V}(t,T,r,\lambda),d\mathbf{M}(t)\rangle\\
+\ &(V(t,T,r,\lambda,x)-V(t,T,r(t^-),\lambda,x))dN(t)\\
+\ &(V(t,T,r,\lambda,x)-V(t,T,r,\lambda(t^-),x))d\overline{N}(t)].
\end{aligned}$$

Recall that $U(t,T,r,\lambda,x)$ is a bounded martingale. Therefore, we have

$$\begin{aligned}
&\frac{\sigma(t)^2}{2}\frac{\partial^2 V}{\partial r^2}+\frac{\partial V}{\partial t}+c\frac{\partial V}{\partial r}(a(t)-r)-\frac{\partial V}{\partial\lambda}b\lambda+\langle Q\mathbf{V}(t,T,r,\lambda),x\rangle\\
+\ &\int_{-\infty}^{\infty}\lambda(V(t,T,r+y,\lambda,x)-V(t,T,r,\lambda,x))dF(y)\\
+\ &\int_0^{\infty}\rho(t)(V(t,T,r,\lambda+z,x)-V(t,T,r,\lambda,x))dH_t(z)=0.
\end{aligned} \tag{3.6}$$

Due to the affine forms of the processes $\boldsymbol{\lambda}$ and $\mathbf{r}$, following Duffie et al. [11], we fit the solution to (3.6) by

$$V(t,T,r,\lambda,x)=e^{A(t,T)r+B(t,T)\lambda}D(t,T,x),$$

with terminal conditions

$$A(T,T)=0,B(T,T)=0,D(T,T,x)=\langle\boldsymbol{\theta},x\rangle.$$

Write

$$\mathbf{D}(t,T)=(D(t,T,e_1),\cdots,D(t,T,e_N))^*.$$

Obviously,

$$D(t,T,x)=\langle\mathbf{D}(t,T),x\rangle.$$

To simplify the notation, we omit the indicator $T$. Then substituting the expression for $V$ into (3.6) yields

$$\begin{aligned} &\frac{d\langle \mathbf{D}(t), x\rangle}{dt} + \langle \mathbf{D}(t), x\rangle [r(\frac{dA}{dt} - cA(t) - 1) + \lambda(\frac{dB}{dt} + \int_{-\infty}^{\infty}(e^{A(t)y} - 1)dF(y) - bB(t)) \\ + \quad &\frac{A^2(t)}{2}\langle \boldsymbol{\sigma}, x\rangle^2 + cA(t)\langle \mathbf{a}, x\rangle + \int_0^{\infty}(e^{B(t)z} - 1)\langle \boldsymbol{\rho}, x\rangle d\langle \mathbf{H}(z), x\rangle] + \langle Q\mathbf{D}(t), x\rangle = 0. \end{aligned} \tag{3.7}$$

Since Eq. (3.7) holds for any $r, \lambda$ and $x = e_i$, the coefficients of $r, \lambda$ and $x$ in the above equation must be zero. It holds that

$$\frac{dA}{dt} - cA(t) - 1 = 0, A(T) = 0, \tag{3.8}$$

$$\frac{dB}{dt} - bB(t) + \int_{-\infty}^{\infty}(e^{A(t)y} - 1)dF(y) = 0, B(T) = 0, \tag{3.9}$$

$$\frac{d\mathbf{D}(t)}{dt} + \Delta(t)\mathbf{D}(t) = 0, \mathbf{D}(T) = \boldsymbol{\theta}, \tag{3.10}$$

where $\Delta(t) = Q + \mathbf{diag}(\mathbf{G}_t)$ with $\mathbf{G}_t = (G_t^1, \cdots, G_t^N)^* \in R^N$ given by (3.5). It is easy to see that the solutions of Eqs. (3.8)-(3.9) are given by (3.2)-(3.3). In order to solve Eq. (3.10), let $\Psi(t)$ denote the fundamental matrix solution to the following linear, matrix-valued, ODE:

$$\frac{d\Psi(t)}{dt} + \Delta(t)\Psi(t) = 0, \Psi(T) = \mathbf{diag}(\boldsymbol{\theta}). \tag{3.11}$$

Since $\Delta(t)$ is continuous, there exists a unique solution of (3.11) over the finite time interval $\mathcal{T}$. Then

$$\mathbf{D}(t) = \Psi(t)\boldsymbol{\theta}.$$

Hence,

$$D(t, x) = \langle \Psi(t)\boldsymbol{\theta}, x\rangle.$$

The proof is finished.

**Corollary 3.1** *Under some suitable integrability conditions, the price of a zero-coupon bond at time $t$ with maturity $T$ is given by*

$$P(t, T) = e^{A(t,T)r(t) + B(t,T)\lambda(t)}\langle \overline{\Psi}(t, T)\mathbf{1}, X(t)\rangle, \tag{3.12}$$

*where $A(t, T), B(t, T)$ are given by (3.2)-(3.3), and $\overline{\Psi}(t, T)$ is determined by (3.4)-(3.5) with $\boldsymbol{\theta}$ replaced by $\mathbf{1} = (1, \cdots, 1)^* \in R^N$.*

*Proof.* Corollary 3.1 is a direct consequence of Theorem 3.1.

Corollary 3.1 shows the stochastic short interest rate we propose allows representation for the exponential affine form of the bond price in terms of a fundamental matrix solution of linear matrix differential equations. Note that Shen and Siu [18] also derived a similar representation for the exponential affine form of the bond price in terms of a fundamental matrix solutions of linear matrix differential equation under a Markovian regime-switching Hull-White model. Our model extends the regime-switching Hull-White model by adding a Markovian regime-switching jump component. Therefore, the bond price in our model is not only affected by the business or economic cycles described by the chain $\mathbf{X}$ but also influenced by sudden jumps of the interest rate dynamics due to some extraordinary market events. Since it is difficult to analytically solve the ODE (3.4)-(3.5), some numerical methods should be employed to find the solution. See for example, we can use a Runge-Kutta algorithm to solve the ODE.

## 4. The aggregate accumulated claims

In this section, we discuss the application of the regime-switching shot noise process (2.3) in insurance. In classical risk theory, the loss process $\mathbf{C} := \{C(t)|t \in \mathcal{T}\}$ is described by

$$C(t) = \sum_{j=1}^{N(t)} Y_j, \tag{4.1}$$

where $\mathbf{N} := \{N(t)|t \in \mathcal{T}\}$ is the number of claims up to time $t$, and $Y_j$ is the amount of the $j$th claim; $\{Y_j; j = 1, 2, \cdots\}$ is assumed to be a sequence of i.i.d. non-negative random variables with the common distribution $F(y)(y > 0)$ satisfying some suitable integrability conditions and it is independent of $\mathbf{N}$. Here we assume that the process

$$\mathbf{N} := \{N(t)|t \in \mathcal{T}\}$$

follows a Cox process with intensity $\boldsymbol{\lambda} := \{\lambda(t)|t \in \mathcal{T}\}$ modelled by (2.3):

$$d\lambda(t) = -b\lambda(t)dt + dL(t), \lambda(0) = \langle \overline{\boldsymbol{\lambda}}, X(0) \rangle.$$

The Markov chain $\mathbf{X} := \{X(t)|t \in \mathcal{T}\}$, which influences the frequencies of the claims, describes the external environment process. As pointed out in [2], in health insurance, sojourns of $\mathbf{X}$ could be certain types of epidemics, or, in automobile insurance, these could be weather types. This particular type of generalization is motivated partly by the flexibility on the modeling of the arrival process, allowing one to model arrival streams that are more irregular than any renewal process, and partly that in some cases, one can interpret the model in a natural way in the sense that an underlying external environment may involve the insurance business (see [3]).

If we consider the effect of the introduction of constant interest rate factors, then the accumulated aggregate claims up to $t$, $\mathbf{S} := \{S(t)|t \in \mathcal{T}\}$, should be defined as

$$S(t) = \sum_{i=1}^{N(t)} Y_i e^{\delta(t-T_i)}, \tag{4.2}$$

where $\delta > 0$ is the risk-free force of interest rate, $T_i$'s are time points at which claims occur.

The process $\mathbf{S}$ also solves the stochastic differential equation

$$dS(t) = \delta S(t)dt + dC(t). \tag{4.3}$$

Note that if $\mathbf{N}$ follows a homogenous Poisson process, then the moments of the aggregate accumulated claims can be found in Jang [16]. In what follows we shall investigate the moments of the accumulated aggregate claims under a Markovian environment.

Let $\mathbb{H}^S := \{\mathcal{H}_t^S|t \in \mathcal{T}\}$ and $\mathbb{H}^\lambda := \{\mathcal{H}_t^\lambda|t \in \mathcal{T}\}$ be the right-continuous, $P$-completed, natural filtrations generated by the processes $\mathbf{S}$ and $\boldsymbol{\lambda}$, respectively. Denote the enlarged filtration by $\overline{\mathbb{H}} := \{\overline{\mathcal{H}}_t|t \in \mathcal{T}\}$, where for each $t \in \mathcal{T}$, $\overline{\mathcal{H}}_t = \mathcal{H}_t^S \vee \mathcal{H}_t^\lambda \vee \mathcal{H}_t^X$.

In order to derive the moments of the aggregate accumulated claims, we define

$$W(t, T, \xi, \eta) = E[e^{-\xi S(T) - \eta\lambda(T)}|\overline{\mathcal{H}}_t],$$

for any $\xi > 0, \eta > 0$.

Note that $(\mathbf{S}, \boldsymbol{\lambda}, \mathbf{X})$ is a three-dimensional Markov process with respect to $\overline{\mathbb{H}}$. Therefore, given $S(t) = s, \lambda(t) = \lambda, X(t) = x$,

$$\begin{aligned} W(t, T, \xi, \eta) &= E[e^{-\xi S(T) - \eta\lambda(T)}|S(t) = s, \lambda(t) = \lambda, X(t) = x] \\ &\doteq \hat{V}(t, T, \xi, \eta, s, \lambda, x). \end{aligned}$$

**Theorem 4.1** *For any $\xi > 0, \eta > 0$, we have*

$$W(t,T,\xi,\eta) = e^{\hat{A}(t,T,\xi)s+\hat{B}(t,T,\xi,\eta)\lambda}\langle\hat{\Psi}(t,T,\xi,\eta)\mathbf{1}, x\rangle, \tag{4.4}$$

*where*

$$\hat{A}(t,T,\xi) = -\xi e^{\delta(T-t)}, \tag{4.5}$$

$$\hat{B}(t,T,\xi,\eta) = -\eta e^{-b(T-t)} + \int_t^T \int_0^\infty e^{-b(s-t)}(e^{\hat{A}(s,T,\xi)y} - 1)dF(y)ds, \tag{4.6}$$

*and $\hat{\Psi}(t,T,\xi,\eta)$ is a fundamental matrix solution to the following linear, matrix-valued, ODE:*

$$\frac{\partial\hat{\Psi}}{\partial t} + \hat{\Delta}(t,T,\xi,\eta)\hat{\Psi}(t,T,\xi,\eta) = 0, \hat{\Psi}(T,T,\xi,\eta) = \mathbf{diag}(\mathbf{1}), \tag{4.7}$$

*with $\hat{\Delta}(t,T,\xi,\eta) = Q + \mathbf{diag}(\hat{\mathbf{G}}_t(\xi,\eta))$, where*

$$\hat{\mathbf{G}}_t(\xi,\eta) = (\hat{G}_t^1(\xi,\eta), \hat{G}_t^2(\xi,\eta), \cdots, \hat{G}_t^N(\xi,\eta))^* \in R^N,$$

*with*

$$\hat{G}_t^i(\xi,\eta) = \rho_i \int_0^\infty (e^{\hat{B}(t,T,\xi,\eta)z} - 1)dH^i(z), i = 1, \cdots, N. \tag{4.8}$$

*Proof.* The proof is similar to that of Theorem 3.1, so we just give an outline. For the simplicity of notation, we will omit the parameters $\xi$ and $\eta$ in $\hat{V}(t,T,\xi,\eta,s,\lambda,x)$. Write

$$\hat{\mathbf{V}}(t,T,s,\lambda) = (\hat{V}(t,T,s,\lambda,e_1), \cdots, \hat{V}(t,T,s,\lambda,e_N))^* \in R^N.$$

Then

$$\hat{V}(t,T,s,\lambda,x) = \langle\hat{\mathbf{V}}(t,T,s,\lambda), x\rangle.$$

Applying Ito's formula to $\hat{V}$ and using the fact that $\hat{V}(t,T,s,\lambda,x)$ is a bounded martingale yield

$$\begin{aligned}
& \frac{\partial\hat{V}}{\partial t} + \frac{\partial\hat{V}}{\partial s}\delta s - \frac{\partial\hat{V}}{\partial\lambda}b\lambda + \langle Q\hat{\mathbf{V}}(t,T,s,\lambda), x\rangle \\
+ & \int_0^\infty \lambda(\hat{V}(t,T,s+y,\lambda,x) - \hat{V}(t,T,s,\lambda,x))dF(y) \\
+ & \int_0^\infty \rho(t)(\hat{V}(t,T,s,\lambda+z,x) - \hat{V}(t,T,s,\lambda,x))dH_t(z) = 0.
\end{aligned}$$

Due to the affine forms of $\boldsymbol{\lambda}$ and $\mathbf{S}$, we fit the solution to the above equation by

$$\hat{V}(t,T,s,\lambda,x) = e^{\hat{A}(t,T)s+\hat{B}(t,T)\lambda}\hat{D}(t,T,x)$$

with terminal conditions

$$\hat{A}(T,T) = -\xi, \hat{B}(T,T) = -\eta, \hat{D}(T,T,x) = 1.$$

Write $\hat{\mathbf{D}}(t,T) = (\hat{D}(t,T,e_1), \cdots, \hat{D}(t,T,e_N))^*$. It is obvious that

$$\hat{D}(t,T,x) = \langle\hat{\mathbf{D}}(t,T), x\rangle.$$

To simplify the notation, we omit the indicator $T$. Then substituting the expression for $\hat{V}$ into the above equation yields

$$\begin{aligned}
& \hat{D}(t,x)(s(\frac{d\hat{A}}{dt} + \delta\hat{A}(t)) + \lambda(\frac{d\hat{B}}{dt} - b\hat{B}(t)) + \frac{d\hat{D}}{dt} + \langle Q\hat{\mathbf{D}}(t), x\rangle \\
+ & \hat{D}(t,x)\int_0^\infty \lambda(e^{\hat{A}(t)y} - 1)dF(y) \\
+ & \hat{D}(t,x)\int_0^\infty (e^{\hat{B}(t)z} - 1)\langle\boldsymbol{\rho}, x\rangle\langle\mathbf{H}(\mathbf{z}), x\rangle dz = 0. \qquad (4.9)
\end{aligned}$$

Then following the same augments for solving (3.7), we can obtain the explicit expressions for $\hat{A}(t,T), \hat{B}(t,T)$ and $\hat{D}(t,T,x)$ from Eq. (4.9). The proof is completed.

The following two results are direct consequences of Theorem 4.1.

**Corollary 4.1** *For $\eta > 0$, the Laplace transform of $\lambda(t)$ is*

$$E[e^{-\eta\lambda(t)}] = e^{-\eta e^{-bt}\lambda(0)}\langle\Phi(0,t,\eta)\mathbf{1}, X(0)\rangle, \tag{4.10}$$

*where $\Phi(s,t,\eta)$ is a fundamental matrix solution to the following linear, matrix-valued, ODE:*

$$\frac{\partial\Phi(s,t,\eta)}{\partial t} + (Q + \mathbf{diag}(\tilde{\mathbf{G}}_s(\eta)))\Phi(s,t,\eta) = 0, \Phi(t,t,\eta) = \mathbf{diag}(\mathbf{1}), \tag{4.11}$$

*with $\tilde{\mathbf{G}}_s(\eta) = (\tilde{G}_s^1(\eta), \tilde{G}_s^2(\eta), \cdots, \tilde{G}_s^N(\eta))^* \in R^N$,*

$$\tilde{G}_s^i(\eta) = \rho_i \int_0^\infty (e^{-\eta e^{-b(t-s)}z} - 1)dH^i(z).$$

**Corollary 4.2** *For $\xi > 0$, the Laplace transform of $S(t)$ is*

$$E[e^{-\xi S(t)}] = e^{\int_0^t e^{-bv}(l_v(\xi)-1)dv\lambda(0)}\langle\phi(0,t,\xi)\mathbf{1}, X(0)\rangle, \tag{4.12}$$

*where*

$$l_v(\xi) = \int_0^\infty e^{-\xi e^{\delta(t-v)}y}dF(y),$$

*and $\phi(s,t,\xi)$ is a fundamental matrix solution to the following linear, matrix-valued, ODE:*

$$\frac{\partial\phi(s,t,\xi)}{\partial t} + (Q + \mathbf{diag}(\tilde{\tilde{\mathbf{G}}}_s(\xi)))\phi(s,t,\xi) = 0, \phi(t,t,\xi) = \mathbf{diag}(\mathbf{1}), \tag{4.13}$$

*with $\tilde{\tilde{\mathbf{G}}}_s(\xi) = (\tilde{\tilde{G}}_s^1(\xi), \tilde{\tilde{G}}_s^2(\xi), \cdots, \tilde{\tilde{G}}_s^N(\xi))^* \in R^N$,*

$$\tilde{\tilde{G}}_s^i(\xi) = \rho_i \int_0^\infty (e^{z\int_s^t e^{-b(v-s)}(l_v(\xi)-1)dv} - 1)dH^i(z).$$

By differentiating the Laplace transforms, we can obtain the means and variances of $\lambda(t)$ and $S(t)$.

**Proposition 4.1** *The expectation of $\lambda(t)$ at time $t$ is given by*

$$E[\lambda(t)] = e^{-bt}\lambda(0) - \langle\varphi(0,t), X(0)\rangle, \tag{4.14}$$

*where $\varphi(0,t) = \lim_{\eta\to 0} \frac{\partial(\Phi(0,t,\eta)\mathbf{1})}{\partial\eta}$.*

*The variance of $\lambda(t)$ at time $t$ is given by*

$$Var[\lambda(t)] = \langle\hat{\varphi}(0,t), X(0)\rangle - \langle\varphi(0,t), X(0)\rangle^2, \tag{4.15}$$

*where $\hat{\varphi}(0,t) = \lim_{\eta\to 0} \frac{\partial^2(\Phi(0,t,\eta)\mathbf{1})}{\partial\eta^2}$.*

**Proof.** If we differentiate Eq. (4.10) with respect to $\eta$ and put $\eta = 0$, then we can easily obtain (4.14). If we differentiate both sides of (4.10) with respect to $\eta$ twice and then let $\eta = 0$, we have

$$E[(\lambda(t))^2] = (e^{-bt}\lambda(0))^2 - 2e^{-bt}\lambda(0)\langle\varphi(0,t), X(0)\rangle + \langle\hat{\varphi}(0,t), X(0)\rangle.$$

Then (4.15) follows from the equality $Var[\lambda(t)] = E[(\lambda(t))^2] - (E[\lambda(t)])^2$.

**Remark 4.1** *(a) If there is no regime switching ($N = 1$), then the process $\boldsymbol{\lambda}$ follows a shot noise process studied in [8, 16], and hence the Laplace transform of $\lambda(t)$ is given by*

$$E[e^{-\eta\lambda(t)}] = e^{-\eta e^{-bt}\lambda(0)+\rho_1\int_0^t\int_0^\infty(e^{-\eta e^{-b(t-s)}z}-1)dH^1(z)ds},$$

*which is consistent with Eq. (7) in [16]. Differentiating the Laplace transform yields*

$$E[\lambda(t)] = \lambda(0)e^{-bt} + \rho_1\overline{p}_1\frac{1-e^{-bt}}{b}, \tag{4.16}$$

*and*

$$Var[\lambda(t)] = \rho_1\overline{m}_1\frac{1-e^{-2bt}}{2b}, \tag{4.17}$$

*where $\overline{p}_1 = \int_0^\infty xdH^1(x) < \infty$, $\overline{m}_1 = \int_0^\infty x^2dH^1(x) < \infty$. Eqs. (4.16) and (4.17) are consistent with Eqs. (16) and (17) in [17].*

*(b)If we let $b \to 0$ and $\rho_i = 0$, for each $i = 1, 2, \cdots, N$, then $\lambda(t) \equiv \lambda(0)$ and the process $\mathbf{N}$ is a homogenous Poisson process. Obviously,*

$$E[\lambda(t)] \equiv \lambda(0),$$

*and*

$$Var[\lambda(t)] = 0.$$

*(c)If we let $b \to 0$ and $\lambda(0) = 0$, then from (2.4) we have $\lambda(t) = \sum_{i=1}^{\overline{N}(t)} Z_j$ is a regime-switching compound Poisson process. Therefore,*

$$E[e^{-\eta\lambda(t)}] = \langle\hat{\Phi}(0,t,\eta)\mathbf{1}, X(0)\rangle,$$

*where $\hat{\Phi}(s,t,\eta)$ is determined by (4.11) with $\tilde{G}_s^i(\eta)$ replaced by $\rho_i\int_0^\infty(e^{-\eta z}-1)dH^i(z)$. From the Laplace transform of $\lambda(t)$, we can obtain*

$$E[\lambda(t)] = -\langle\varsigma(0,t), X(0)\rangle,$$

*and*

$$Var[\lambda(t)] = \langle\hat{\varsigma}(0,t), X(0)\rangle - \langle\varsigma(0,t), X(0)\rangle^2,$$

*where $\varsigma(0,t) = \lim_{\eta\to 0}\frac{\partial(\hat{\Phi}(0,t,\eta)\mathbf{1})}{\partial\eta}$, and $\hat{\varsigma}(0,t) = \lim_{\eta\to 0}\frac{\partial^2(\hat{\Phi}(0,t,\eta)\mathbf{1})}{\partial\eta^2}$.*

*In particular, when $N = 1$, the process $\boldsymbol{\lambda}$ becomes a compound Poisson process and hence*

$$E[e^{-\eta\lambda(t)}] = e^{\rho_1(\int_0^\infty e^{-\eta z}dH^1(z)-1)}.$$

*Then it is easy to obtain*

$$E[\lambda(t)] = \rho_1\overline{p}_1,$$

*and*

$$Var[\lambda(t)] = \rho_1\overline{m}_1.$$

**Proposition 4.2** *The expectation of $S(t)$ at time $t$ is given by*

$$E[S(t)] = \frac{\lambda(0)p(e^{\delta t}-e^{-bt})}{b+\delta} - \langle\vartheta(0,t), X(0)\rangle, \tag{4.18}$$

*where $p = \int_0^\infty xdF(x) < \infty$, $\vartheta(0,t) = \lim_{\xi\to 0}\frac{\partial(\phi(0,t,\xi)\mathbf{1})}{\partial\xi}$.*

*The variance of $S(t)$ at time $t$ is given by*

$$Var[S(t)] = \frac{\lambda(0)m(e^{2\delta t} - e^{-bt})}{b+2\delta} + \langle\hat{\vartheta}(0,t), X(0)\rangle - \langle\vartheta(0,t), X(0)\rangle^2, \tag{4.19}$$

*where* $m = \int_0^\infty x^2 dF(x) < \infty$, $\hat{\vartheta}(0,t) = \lim_{\xi\to 0} \frac{\partial^2(\phi(0,t,\xi)\mathbf{1})}{\partial\xi^2}$.

**Proof.** Eq. (4.18) can be obtained by differentiating Eq. (4.12) with respect to $\xi$ and putting $\xi = 0$. If we differentiate Eq. (4.12) with respect to $\xi$ twice and put $\xi = 0$, then we can get

$$E[(S(t))^2] = \langle\hat{\vartheta}(0,t), X(0)\rangle - 2\langle\vartheta(0,t), X(0)\rangle\frac{\lambda(0)p(e^{\delta t} - e^{-bt})}{b+\delta} + \frac{\lambda(0)m(e^{2\delta t} - e^{-bt})}{b+2\delta}.$$

Therefore, Eq. (4.19) follows from the equality $Var[S(t)] = E[(S(t))^2] - (E[S(t)])^2$.

**Remark 4.2** *(a)If there is no regime switching ($N = 1$), then the Laplace transform of $S(t)$ is*

$$E[e^{-\xi S(t)}] = e^{\lambda(0)\int_0^t e^{-bv}(l_v(\xi)-1)dv + \int_0^t \tilde{\tilde{G}}_s^1(\xi)ds},$$

*where $l_v(\xi)$ and $\tilde{\tilde{G}}_s^1(\xi)$ are defined in Corollary 4.2. Note that if we replace the positive constant $\delta$ by $-\delta$, Dassios and Jang [9] called the process $(\boldsymbol{\lambda}, \mathbf{S})$ a double shot noise process. Differentiating the Laplace transform $S(t)$ gives*

$$E[S(t)] = \frac{\lambda(0)p(e^{\delta t} - e^{-bt})}{b+\delta} + \rho_1 p\overline{p}_1(\frac{e^{\delta t}-1}{b\delta} - \frac{e^{\delta t} - e^{-bt}}{b(b+\delta)}), \tag{4.20}$$

*and*

$$\begin{aligned} Var[S(t)] &= \frac{\lambda(0)m(e^{2\delta t} - e^{-bt})}{b+2\delta} + \rho_1 m\overline{p}_1(\frac{e^{2\delta t}-1}{2b\delta} - \frac{e^{2\delta t} - e^{-bt}}{b(b+2\delta)}) \\ &+ \rho_1\overline{m}_1 p^2(\frac{e^{2\delta t}-1}{2\delta} + \frac{1-e^{-2bt}}{2b} - \frac{2(1-e^{-(b-\delta)t})}{b-\delta}), \end{aligned} \tag{4.21}$$

*where $p, m, \overline{p}_1, \overline{m}_1$ are given in Proposition 4.2 and Remark 4.1.*

*If we multiply $e^{-\delta t}$ by $S(t)$, then from (4.20) and (4.21) we can obtain the expectation and the variance of the discounted aggregated claims $S^0(t) := e^{-\delta t}S(t)$ :*

$$E[S^0(t)] = \frac{\lambda(0)p(1 - e^{-(\delta+b)t})}{b+\delta} + \rho_1 p\overline{p}_1(\frac{1-e^{-\delta t}}{b\delta} - \frac{1 - e^{-(b+\delta)t}}{b(b+\delta)}), \tag{4.22}$$

*and*

$$\begin{aligned} Var[S^0(t)] &= \frac{\lambda(0)m(1 - e^{-(b+2\delta)t})}{b+2\delta} + \rho_1 m\overline{p}_1(\frac{1-e^{-2\delta t}}{2b\delta} - \frac{e^{2\delta t} - e^{-bt}}{b(b+2\delta)}) \\ &+ \rho_1\overline{m}_1 p^2(\frac{1-e^{-2\delta t}}{2\delta} + \frac{e^{-2\delta t} - e^{-2(b+\delta)t}}{2b} - \frac{2(e^{-2\delta t} - e^{-(b+\delta)t})}{b-\delta}), \end{aligned} \tag{4.23}$$

*It is easy to check that Eqs. (4.22) and (4.23) are consistent with Eqs. (33) and (43) in [9].*

*(b)If we let $b \to 0$ and $\rho_i = 0$, for each $i = 1, 2, \cdots, N$, then the loss process $\mathbf{C}$ becomes a compound Poisson process and the process $\mathbf{S}$ becomes a classic accumulated aggregate claims process. Hence,*

$$E[e^{-\xi S(t)}] = e^{\lambda(0)\int_0^t\int_0^\infty(e^{-\xi e^{\delta(t-v)}y}-1)dF(y)dv}.$$

*Note that the expression for $E[e^{-\xi S(t)}]$ in this case is similar to the formula for $E[e^{-\eta\lambda(t)}]$ when $\boldsymbol{\lambda}$ follows a shot noise process. This is because if we replace the positive constant $\delta$ by $-\delta$, then the process $\mathbf{S}$ also follows a shot noise process. Then from the above Laplace transform, we can obtain*

$$E[S(t)] = \frac{\lambda(0)p(e^{\delta t}-1)}{\delta}, \tag{4.24}$$

*and*

$$Var[S(t)] = \frac{\lambda(0)m(e^{2\delta t} - 1)}{2\delta}. \tag{4.25}$$

**5. Conclusions**

This paper develops a Markov regime-switching model with jumps, where the jumps are described by a compound Cox process with the intensity process modelled by a regime-switching shot noise process. Under the proposed model, we study the bond pricing and the moments of aggregate accumulated claims considering the changes of market regimes and environments. By using a martingale method, we derive exponential-affine form expressions for the bond price and the joint Laplace transform of the aggregate accumulated claims and the arrival intensity.

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